Designing routes for vehicles and drivers

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VeRoLog EURO Working Group on Vehicle Routing and Logistics Optimization

Contents

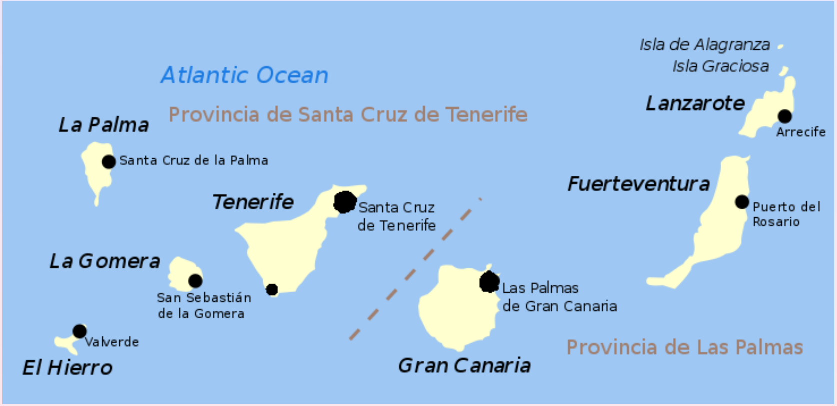
Bad time for regional airline companies; closed in Spain since 2000:

- Galicia. Prima Air
- Navarra. Líneas Aéreas Navarras (Air Truck)
- Málaga. Pauknair
- Vitoria. ERA
- Mallorca. Air Europa Express
- Málaga. Binter Mediterráneo
- Baleares. Aebal (closed on September 2008)
- La Rioja. Iberline
- Barcelona. Air Catalunya. Intermed
- Asturias. Air Asturias (closed on January 2007)
- La Rioja. Rioja Airlines (closed on 9 September 2007)
- León. Lagun Air (closed on January 2005)
- Córdoba. Flysur (Taer Andalus) (closed on 9 October 2008)
- Almería. Ándalus (closed on June 2010)
- Granada. Helitt (closed on 23 September 2014)
- Canarias. Top Fly (closed on Nov 2009). Islas Airways (closed on August 2014)

Not mentioning larger airlines in Spain like: LTE, Hola Airlines, Bravo Airlines, Quantum Air, Spantax, Volar Airlines, Tadair, Futura, Air Madrid, Air Comet, Spanair.
Regional airlines operating today in Spain: Air Nostrum, **Binter Canarias**, CanaryFly.


Andorra
España
Sitges
Portugal
Lisboa
Gibraltar
Rabat
Madeira
Funchal
Marruecos
Marrakech
Canary Islands
El Aaiun

One geographical region, 2 provinces, 8 airports



The company: BinterCanarias (www.bintercanarias.com)

- 11 airports: LPA, TFN, TFS, SPC, VDE, ACE, FUE, GMZ ; FNC, EUN, RAK
- 18 aircrafts ATR 72

- around 150 daily flights from 07:00 to 23:00
- It moves 80% for the air transport between Canary Islands
- It transported around 2,600,000 persons in 2014
- aircrafts and crews are divided in 3 operators: Binter,Naysa,Canair
  Binter has 2 aircrafts, Naysa has 12 aircrafts, Canair has 4 aircrafts.
- crews of each operator are divided in 2 bases depending on the home island (TFN , LPA)

Traditional optimization problems for airline companies are:

- Flight scheduling
- Fleet assignment
- Aircraft routing
- Crew pairing
- Crew rostering
- Aircraft rostering
- Disruption management

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We solve the first 4 problems (related to a given day) in a **single integrated routing problem**, for our regional airline company, as follows:

Given a fixed day, our routing problem is

The *customers* are the flights (around 150), each defined by

- flight number; example: 104
- departure airport; example: TFN
- departure time; example: 07:30
- arrival airport; example: LPA
- arrival time; example: 08:00

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- departure time; example: 07:30
- arrival airport; example: LPA
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The *depots* are the two special airports (bases): TFN & LPA

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- flight number; example: 104
- departure airport; example: TFN
- departure time; example: 07:30
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The *depots* are the two special airports (bases): TFN & LPA

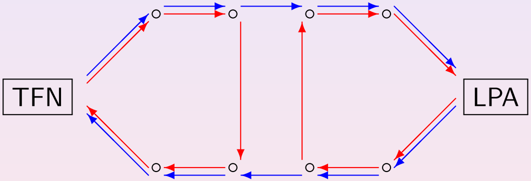
The *vehicles* are: crews & aircrafts

We will talk here about designing 2 type of routes:

- routes for crews
- routes for aircrafts

Requirements:

- Each crew must sleep in its base: TFN or LPA.
- Each aircraft must sleep alternatively inside and outside LPA.



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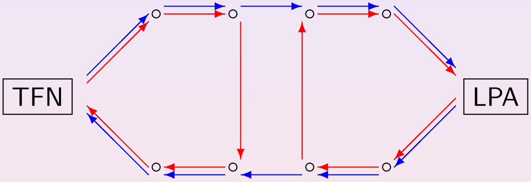

⇒ Aircraft Changes: some crews change from one aircraft to another aircraft

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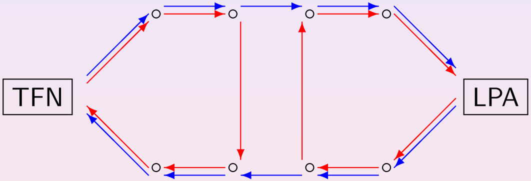
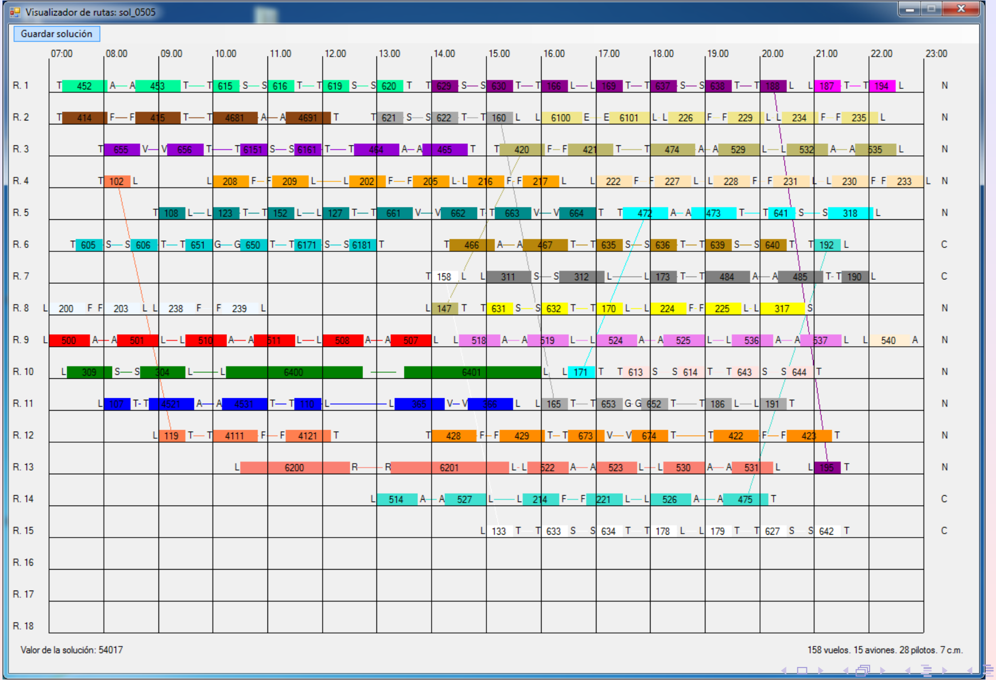


⇒ Aircraft Changes: some crews change from one aircraft to another aircraft

A crew cannot operate more than 8 flights (per day) and cannot work more than 9 consecutive hours (per day)


Visualizador de rutas: sol_0505
Guardar solución
07:00 08.00 09.00 10.00 11.00 12.00 13.00 14.00 15.00 16.00 17.00 18.00 19.00 20.00 21.00 22.00 23:00
R. 1
R. 2
R. 3
R. 4
R. 5
R. 6
R. 7
R. 8
R. 9
R. 10
R. 11
R. 12
R. 13
R. 14
R. 15
R. 16
R. 17
R. 18
Valor de la solución: 54017
158 vuelos. 15 aviones. 28 pilotos. 7 c.m.

- Airline Scheduling (aircraft routing, crew pairing)
  - Cordeau, Stojkovic, Soumis, Desrosiers (2001) in TS
  - Klabjan, Johnson, Nemhauser, Gelman, Ramaswamy (2002) in TS
  - Cohn, Barnhart (2003) in OR
  - Mercier, Cordeau, Soumis (2005) in C&OR
  - Mercier, Soumis (2007) in C&OR
  - Weide, Ryan, Ehrgott (2010) in C&OR
  - ...
- Vehicle Routing Problem with Time Windows
  - Bard, Kontoravdis, Yu (2002) in OR
  - Kallehauge, Boland, Madsen (2007) in N
  - Ropke, Cordeau, Laporte (2007) in N
  - Jepsen, Petersen, Spoorendonk, Pisinger (2008) in OR
  - Kallehauge (2008) in C&OR
  - ...

2-depot driver-and-vehicle routing problem: arc model

$V := V_d \cup V_c$ where $V_d := \{\text{TFN}, \text{LPA}\}$ and V_c are flights

$x_{ij} = 1$ if and only if a crew will fly j immediately after i
$y_{ij} = 1$ if and only if an aircraft will fly j immediately after i
$z_{ij} = 1$ if and only if an "aircraft change" will occur from i to j

$$\min \sum_{i,j \in V_c} c_{ij} x_{ij} + \alpha \sum_{d \in V_d} x(\delta^+(d)) + \beta \sum_{d \in V_d} y(\delta^+(d)) + \gamma \sum_{i,j \in V_c} z_{ij}$$

$$x(\delta^+(i)) = x(\delta^-(i)) \quad [= 1 \text{ if } i \in V_c] \quad \text{for all } i \in V \tag{1}$$

$$x(A(S)) \leq |S| - 1 \quad \text{for all } S \subseteq V_c \tag{2}$$

$$x(d_1 : S) + x(A(S)) + x(S : d_2) \leq |S| \quad \text{for all } S \subseteq V_c, \{d_1, d_2\} = V_d \tag{3}$$

$$y(\delta^+(i)) = y(\delta^-(i)) \quad [= 1 \text{ if } i \in V_c] \quad \text{for all } i \in V \tag{4}$$

$$y(A(S)) \leq |S| - 1 \quad \text{for all } S \subseteq V_c \tag{5}$$

$$y(d : S) + y(A(S)) + y(S : d) \leq |S| \quad \text{for all } S \subseteq V_c, d \in V_d \tag{6}$$

$$x_{ij} \leq y_{ij} + z_{ij} \quad \text{for all } i,j \in V_c \tag{7}$$

"syncronization" between crew and aircraft (8)

Time-synchronization constraints

t_i = elapsed time of serving customer i
t_{ij} = elapsed time of moving a driver or vehicle from i to j
T = time period (big-M)
Example: $t_i = 30$ minutes, $t_{ij} = 30$ minutes, $T = 16$ hours.

w_i = [unknown] wall-clock time when customer i starts being served

For all $i, j \in V_c$:

$$w_j \geq w_i + (t_i + t_{ij})x_{ij} - (T - t_i)(1 - x_{ij}) \tag{9}$$
$$w_j \geq w_i + (t_i + t_{ij})y_{ij} - (T - t_i)(1 - y_{ij}) \tag{10}$$

When flight departures are given, i.e. no schedule redesign

Customer i requires to start being served at a **precise** time.

A crew and aircraft can go from i to j only if

- the arrival airport of i is the departure airport of j
- the arrival time of i $(+t_{ij})$ is before the departure time of j

Then the underlying graph is asymmetric and **acyclic**.

On our real-world instances: about 150 nodes and about 1500 arcs.

and the exponential-number of SECs (2)–(3) and (5)–(6), and variables w_i are useless.

aircraft routing & crew pairing & little schedule redesign

When flight retiming are allowed to potentially have better (or feasible) routing solutions:
Each flight is given with a proposal of departure time and with plus/minus a possible modification.
For example, flight i will departure at 09:15 within $+10$ or -10 minutes.

aircraft routing & crew pairing & little schedule redesign

When flight retiming are allowed to potentially have better (or feasible) routing solutions:
Each flight is given with a proposal of departure time and with plus/minus a possible modification.
For example, flight i will departure at 09:15 within $+10$ or -10 minutes.

Then w_i is a necessary variable assuming values in $[e_i, l_i]$.
In our case $l_i - e_i < t_{ij}$, thus the graph is *almost* acyclic.
For each i and j, there is an arc (i,j) if $e_i + t_i + t_{ij} \le l_j$, and

$$w_j \ge w_i + (t_i + t_{ij})x_{ij} + (e_j - l_i)(1 - x_{ij})$$
$$w_j \ge w_i + (t_i + t_{ij})y_{ij} + (e_j - l_i)(1 - y_{ij})$$

Thus big-M values are not so problematic!

aircraft routing & crew pairing & discrete retiming

When a flight has a set of 3 to 5 options for departure, we add a node to the graph representing each flight and each potential departure.

aircraft routing & crew pairing & discrete retiming

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- H.D. Sherali, K-H. Bae, and M. Haouari. An integrated approach for airline ight selection and timing, fleet assignment, and aircraft routing. Transportation Science, 47(4):455–476, 2013.
- R. van Lieshout, J. Mulder, and D. Huisman. The vehicle rescheduling problem with retiming. Computers & Operations Research, 96:131–140, 2018.
- V. Cacchiani, J-J. Salazar-González. Heuristic approaches for flight retiming in an integrated airline scheduling problem of a regional carrier. Omega (to appear) 2019.

Master problem:

$$\min \sum_{k=1}^{m} \sum_{P \in \mathcal{P}^k} c_P x_P + \alpha \sum_{k=1}^{m} \sum_{P \in \mathcal{P}^k} x_P + \beta \sum_{h=1}^{r} \sum_{Q \in \mathcal{Q}^h} y_Q + \gamma \sum_{a \in A} z_a,$$

$$\begin{aligned}
\sum_{P \in \mathcal{P}^k} x_P &\le f^k, && k = 1, \dots, m, \\
\sum_{k=1}^{m} \sum_{P \in \mathcal{P}^k_j} x_P &= 1, && j \in N^C, \\
\sum_{Q \in \mathcal{Q}^h} y_Q &\le g^h, && h = 1, \dots, r, \\
\sum_{h=1}^{r} \sum_{Q \in \mathcal{Q}^k_j} y_Q &= 1, && j \in N^C, \\
\sum_{k=1}^{m} \sum_{P \in \mathcal{P}^k_a} x_P \le \sum_{h=1}^{r} \sum_{Q \in \mathcal{Q}^h_a} y_Q &+ z_a, && a \in A, \\
x_P &\in \{0,1\}, && k = 1, \dots, m, \; P \in \mathcal{P}^k, \\
y_Q &\in \{0,1\}, && h = 1, \dots, r, \; Q \in \mathcal{Q}^h, \\
z_a &\in \{0,1\}, && a \in A.
\end{aligned}$$

2-depot driver-and-vehicle routing problem: path model

...

Subproblem:

- Maximum-profit path P from airport σ_c^k to airport τ_c^k in (N, A) with node profits $\bar{\beta}_j$ for each $j \in N^C$ and arc profits $\bar{\psi}_a$ for each $a \in A$, including at most 8 nodes and duration at most 9 hours.
- Maximum-profit path Q from airport σ_a^h to airport τ_a^h in (N, A) with node profits $\bar{\phi}_j$ for each $j \in N^C$ and arc profits $\bar{\psi}_a$ for each $a \in A$.

Both path problems can be solved by a dynamic-programming procedure.

2-depot driver-and-vehicle routing problem: path model

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Subproblem:

- Maximum-profit path P from airport σ_c^k to airport τ_c^k in (N, A) with node profits $\bar{\beta}_j$ for each $j \in N^C$ and arc profits $\bar{\psi}_a$ for each $a \in A$, including at most 8 nodes and duration at most 9 hours.
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Both path problems can be solved by a dynamic-programming procedure.

Still, the best model uses path-variables for the crew aspect and arc-variables for the aircraft aspect.

Arc-path method (versus Path-path method)

Algorithm based on the model with path-variables for crews and arc-variables for aircrafts:

Step 1: Solve the linear-programming relaxation: LB

Step 2: Solve ($<$2 minute) the relaxation with integer variables: UB

Step 3: Generate all path-variables with reduce cost smaller than UB$-$LB

Step 4: Solve ($<$2 hour) the extended model with integer variables: OPT

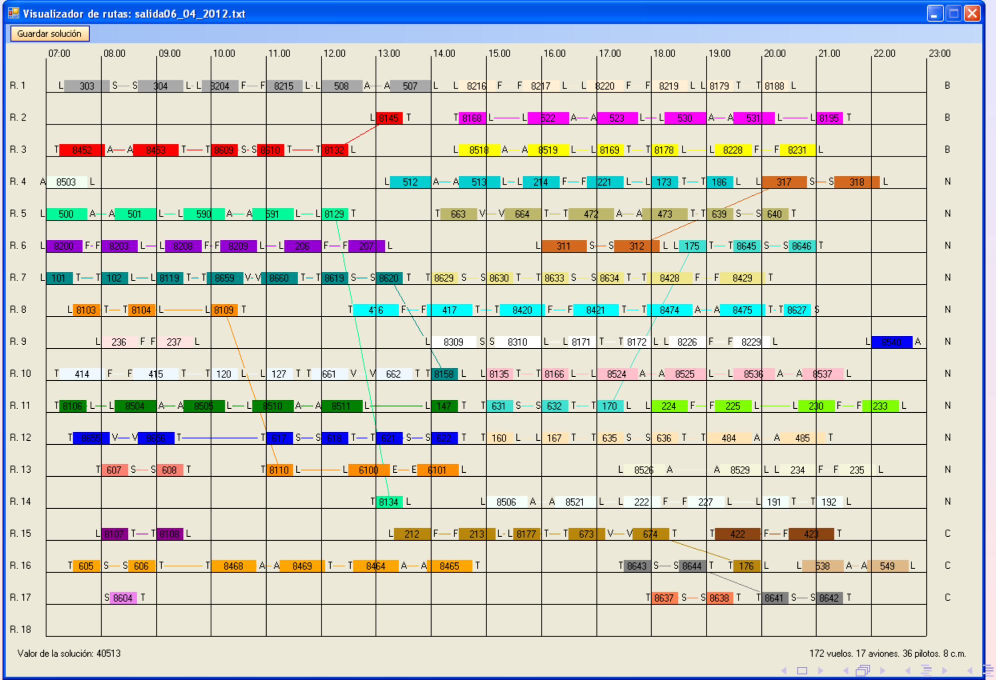
Branch-and-cut framework: CPLEX 12.4

Computer: CORE i5-2400 3.10GHZ, 16GB RAM

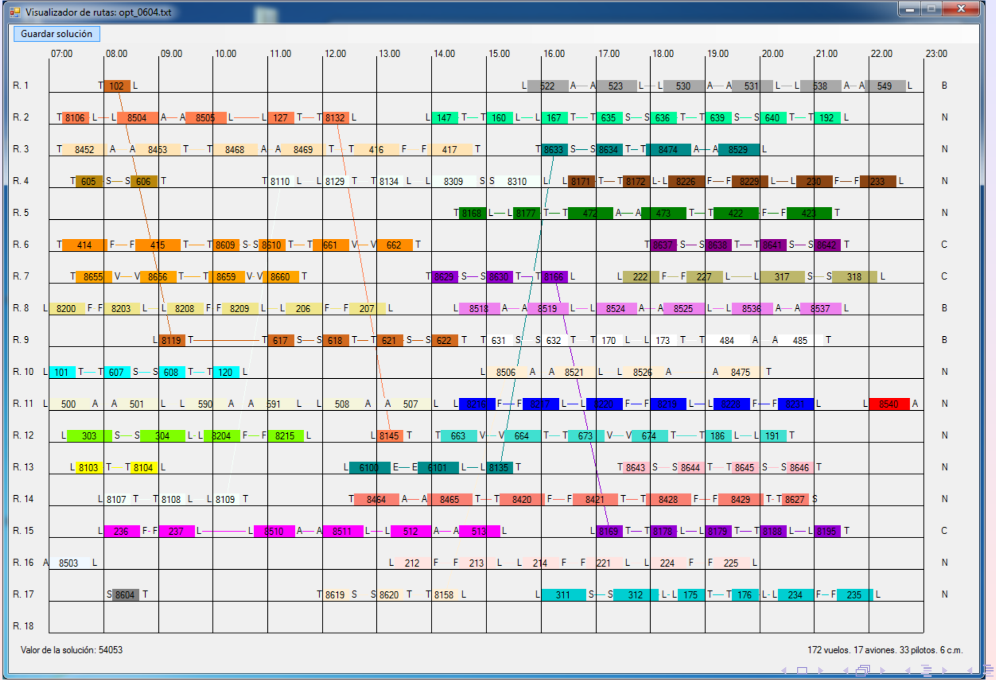
		Path-path method					Arc-path method				
Inst.	#f	%Gap	T_{LB}	T_{UB}	BEST	T_{Tot}	%Gap	T_{LB}	T_{UB}	BEST	T_{Tot}
1/9	102	0.45	42	120	22028	2868	0.09	31	22	22028	551
2/9	140	-	218	120	28886	TL	0.07	148	13	27828	315
3/9	130	-	132	120	25172	3228	0.02	108	5	25172	136
4/9	124	-	101	120	24018	TL	0.08	73	120	24017	2088
5/9	124	-	116	120	24122	TL	0.08	89	62	24121	664
6/9	128	0.40	134	120	24796	TL	0.08	94	120	24796	1777
7/9	150	-	286	120	28049	TL	0.08	255	120	27970	6589
1/4	138	-	231	120	27288	TL	0.07	223	120	27287	7068
2/4	132	-	150	120	28985	693	0.00	130	4	28985	134
3/4	138	-	172	120	29266	1241	0.23	156	120	29266	286
4/4	136	-	162	120	29101	724	0.00	188	6	29101	194
5/4	144	-	207	120	29962	842	0.00	223	22	29962	245
6/4	172	-	640	120	33103	1983	0.00	653	29	33103	682
7/4	100	-	32	120	21300	TL	0.44	31	34	21300	498

		Path-path model				Arc-path model						
Inst.	#f	$x_R$lp	$y_R$lp	$x_R$bp	$y_R$bp	$x_R$lp	$y_a$lp	$x_R$gap	$y_a$gap	Tx_R	Ty_R	Ty_a
1/9	102	1259	1576	2337	7069	654	10128	537	3779	10838	1065025	10128
2/9	140	2195	2773	7592	19383	997	17609	1061	6130	34643	6603705	17609
3/9	130	1731	2229	3575	9203	839	16346	516	4206	30398	5989657	16346
4/9	124	1493	1995	5300	13592	754	15005	827	6281	23048	3843521	15005
5/9	124	1528	2142	5061	14712	811	14985	658	5361	22896	3848204	14985
6/9	128	1674	2274	4243	17123	785	15676	843	5679	25566	4570737	15676
7/9	150	2271	2848	6335	15515	1119	21511	1030	8434	60910	16501692	21511
1/4	138	2578	2873	24519	11349	1092	18892	1647	6882	73120	14196210	18892
2/4	132	1988	2329	2220	3115	908	19071	-	-	63804	18144693	19071
3/4	138	2006	2308	2611	4285	861	21095	5192	18606	84627	23753712	21095
4/4	136	2024	2396	2193	3079	957	20744	-	-	83098	23425413	20744
5/4	144	2190	2631	2317	3330	999	22903	-	-	102195	31424952	22903
6/4	172	3219	3695	3517	4824	1321	30473	-	-	207776		30473
7/4	100	1179	1370	4918	10504	709	10786	2563	9873	15411	1979296	10786

		Arc-path method				Manual solution				
Inst.	#f	#a	#c	#ch	Opt	#a	#c	#ch	Heu	%Gap
1/9	102	12	22	4	22028	14	24	8	24600	11.68
2/9	140	15	28	5	27828	17	28	7	29045	4.37
3/9	130	14	25	5	25172	15	26	6	26818	6.54
4/9	124	13	24	4	24017	13	25	5	25450	5.97
5/9	124	13	24	4	24121	14	25	4	25466	5.58
6/9	128	13	25	4	24796	13	26	4	26368	6.34
7/9	150	15	28	5	27970	15	30	5	30127	7.71
1/4	138	17	27	6	27287	17	28	7	29163	6.88
2/4	132	16	29	6	28985	16	31	6	31467	8.56
3/4	138	16	29	6	29266	16	31	6	32273	10.27
4/4	136	16	29	6	29101	16	35	6	35956	23.56
5/4	144	16	30	6	29962	16	31	6	31798	6.13
6/4	172	17	33	6	33103	17	36	8	36341	9.78
7/4	100	12	21	4	21300	11	24	4	24622	15.60



Visualizador de rutas: salida06_04_2012.txt
Guardar solución
07:00 08.00 09.00 10.00 11.00 12.00 13.00 14.00 15.00 16.00 17.00 18.00 19.00 20.00 21.00 22.00 23:00
R. 1 303 304 8204 8215 508 507 8216 8217 8220 8219 8179 8188 B
R. 2 8145 8168 522 523 530 531 8195 B
R. 3 8452 8453 8609 8610 8132 8518 8519 8169 8178 8228 8231 B
R. 4 8503 512 513 214 221 173 186 317 318 N
R. 5 500 501 590 591 8129 663 664 472 473 639 640 N
R. 6 8200 8203 8208 8209 206 207 311 312 175 8645 8646 N
R. 7 101 102 8119 8659 8660 8619 8620 8629 8630 8633 8634 8428 8429 N
R. 8 8103 8104 8109 416 417 8420 8421 8474 8475 8627 N
R. 9 236 237 8309 8310 8171 8172 8226 8229 8540 N
R. 10 414 415 120 127 661 662 8158 8135 8166 8524 8525 8536 8537 N
R. 11 8106 8504 8505 8510 8511 147 631 632 170 224 225 230 233 N
R. 12 8655 8656 617 618 621 622 160 167 635 636 484 485 N
R. 13 607 608 8110 6100 6101 8526 8529 234 235 N
R. 14 8134 8506 8521 222 227 191 192 N
R. 15 8107 8108 212 213 8177 673 674 422 423 C
R. 16 605 606 8468 8469 8464 8465 8643 8644 176 538 549 C
R. 17 8604 8637 8638 8641 8642 C
R. 18
Valor de la solución: 40513
172 vuelos. 17 aviones. 36 pilotos. 8 c.m.



Visualizador de rutas: opt_0604.txt
Guardar solución
07:00 08.00 09.00 10.00 11.00 12.00 13.00 14.00 15.00 16.00 17.00 18.00 19.00 20.00 21.00 22.00 23:00
R. 1 T 102 L L 522 A A 523 L L 530 A A 531 L L 538 A A 549 L B
R. 2 T 8106 L L 8504 A A 8505 L L 127 T T 8132 L L 147 T T 160 L L 167 T T 635 S S 636 T T 639 S S 640 T T 192 L N
R. 3 T 8452 A A 8453 T T 8468 A A 8469 T T 416 F F 417 T T 8633 S S 8634 T T 8474 A A 8529 L N
R. 4 T 605 S S 606 T T 8110 L L 8129 T T 8134 L L 8309 S S 8310 L L 8171 T T 8172 L L 8226 F F 8229 L L 230 F F 233 L N
R. 5 T 8168 L L 8177 T T 472 A A 473 T T 422 F F 423 T N
R. 6 T 414 F F 415 T T 8609 S S 8610 T T 661 V V 662 T T 8637 S S 8638 T T 8641 S S 8642 T C
R. 7 T 8655 V V 8656 T T 8659 V V 8660 T T 8629 S S 8630 T T 8166 L L 222 F F 227 L L 317 S S 318 L C
R. 8 L 8200 F F 8203 L L 8208 F F 8209 L L 206 F F 207 L L 8518 A A 8519 L L 8524 A A 8525 L L 8536 A A 8537 L B
R. 9 L 8119 T T 617 S S 618 T T 621 S S 622 T T 631 S S 632 T T 170 L L 173 T T 484 A A 485 T B
R. 10 L 101 T T 607 S S 608 T T 120 L L 8506 A A 8521 L L 8526 A A 8475 T N
R. 11 L 500 A A 501 L L 590 A A 591 L L 508 A A 507 L L 8216 F F 8217 L L 8220 F F 8219 L L 8228 F F 8231 L L 8540 A N
R. 12 L 303 S S 304 L L 8204 F F 8215 L L 8145 T T 663 V V 664 T T 673 V V 674 T T 186 L L 191 T N
R. 13 L 8103 T T 8104 L L 6100 E E 6101 L L 8135 T T 8643 S S 8644 T T 8645 S S 8646 T N
R. 14 L 8107 T T 8108 L L 8109 T T 8464 A A 8465 T T 8420 F F 8421 T T 8428 F F 8429 T T 8627 S N
R. 15 L 236 F F 237 L L 8510 A A 8511 L L 512 A A 513 L L 8169 T T 8178 L L 8179 T T 8188 L L 8195 T C
R. 16 A 8503 L L 212 F F 213 L L 214 F F 221 L L 224 F F 225 L N
R. 17 S 8604 T T 8619 S S 8620 T T 8158 L L 311 S S 312 L L 175 T T 176 L L 234 F F 235 L N
R. 18
Valor de la solución: 54053
172 vuelos. 17 aviones. 33 pilotos. 6 c.m.

notation for crew-rostering

I= set of crews ; J= set of days ; K= set of crew routes

Input:

$$a_{ij} = \begin{cases} 1 & \text{if crew } i \text{ is available on day } j \\ 0 & \text{otherwise.} \end{cases}$$

$$d_i = \text{number of working days for crew } j \text{ (e.g. 15 if he works 50\% part time)}$$

Output:

$$x_{ijk} = \begin{cases} 1 & \text{if crew } i \text{ flies in day } j \text{ following route } k \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if crew } i \text{ starts 3-day holidays in day } j \\ 0 & \text{otherwise.} \end{cases}$$

Other variables

$$
\begin{aligned}
nM_i &= \text{number of morning routes for crew } i \\
nT_i &= \text{number of afternoon routes for crew } i \\
ztime &= \text{longest working time} \\
zM &= \text{worst number of morning routes} \\
zT &= \text{worst number of afternoon routes} \\
zroutes &= \text{worst number of (real and fictitious) routes} \\
zimag &= \text{worst number of fictitious routes} \\
zpern &= \text{worst number of non-cycle routes}
\end{aligned}
$$

$$\min \alpha \cdot ztime + \beta \cdot zM + \gamma \cdot zT + \delta \cdot zroutes + \delta \cdot zimg + \epsilon \cdot zpern$$

$$\sum_i x_{ijk} = 1 \qquad \text{for all } j, k$$

$$\sum_k x_{ijk} + y_{ij} + y_{i,j-1} + y_{i,j-2} \leq a_{ij} \qquad \text{for all } i, j$$

$$\sum_{j=F,S} y_{ij} = 1 \qquad \text{for all } i$$

$$\sum_k x_{i,j-1,k} \geq y_{ij} \qquad \text{for all } i, j$$

$$\sum_k x_{i,j+3,k} \geq y_{ij} \qquad \text{for all } i, j$$

$$\sum_k \sum_{l=0}^{MaxD} x_{i,j+l,k} \leq MaxD \qquad \text{for all } i, j$$

$$\ldots$$

$$\sum_{k=K^T(j)} x_{ijk} + \sum_{k=K^M(j+1)} x_{i,j+1,k} \leq 1 \qquad \text{for all } i,j$$

$$\sum_{k=K^T(j)} x_{ijk} + (1 - \sum_{k\in K(j+1)} x_{i,j+1,k}) + \sum_{k=K^M(j+2)} x_{i,j+2,k} \leq 2 \qquad \text{for all } i,j$$

$$\sum_{k=0^T(j)} x_{ijk} + (1 - \sum_{k\in K(j+1)} x_{i,j+1,k}) + \sum_{k=0^M(j+2)} x_{i,j+2,k} \leq 2 \qquad \text{for all } i,j$$

$$\sum_{j}\sum_{k} x_{ijk} \leq \sum_{j} a_{ij} - \frac{d_i}{dmax} freeD \qquad \text{for all } i$$

$$\sum_{j}\sum_{k\in K^M(j)} x_{ijk} = nM_i \qquad \text{for all } i$$

$$\sum_{j}\sum_{k\in K^T(j)} x_{ijk} = nT_i \qquad \text{for all } i$$

$$-maxDMT \leq nM_i - nT_i \leq maxDMT \qquad \text{for all } i$$

$$nM_i \leq \frac{d_i}{dmax} zM \qquad \text{for all } i$$

$$nT_i \leq \frac{d_i}{dmax} zT \qquad \text{for all } i$$

$$\sum_j \sum_k t_k \cdot x_{ijk} \leq \frac{d_i}{dmax} ztime \qquad \text{for all } i$$

Computational Results

$\|I\|$	$\|K\|$	nvars	ncons	LP	tLP	LB	UB	gap
7	96	904	1321	4969.0	0.3	4969.0	4970	0.02
10	177	2098	2547	6608.0	1.0	6608.0	6615	0.10
26	478	12254	14895	4683.7	18.7	4683.7	4770	1.84
18	300	4652	6004	3814.4	3.2	3814.4	3840	0.67
7	100	918	1323	4969.0	0.3	4969.0	4970	0.02
13	179	2738	3259	5084.0	1.4	5084.0	5100	0.31
23	487	10680	10000	5339.9	14.5	5339.9	5530	3.56
17	293	4326	5679	4066.4	3.0	4066.4	4085	0.46
16	126	2056	2896	4606.6	0.7	4606.6	4625	0.40
23	202	4815	5628	5873.7	1.7	5873.7	5935	1.04
47	619	20782	26542	5562.6	9.0	5562.6	5980	7.50
30	374	7688	9854	4766.2	4.5	4766.2	4890	2.60

SCIP 3.0 + Soplex 1.7 , Intel Core Duo 2.17 Ghz , time limit =1 hour.

Visualizador de turnos: N_Abril_CMTE_LPA.txt

Martes 24/04/2012	Miercoles 25/04/2012	Jueves 26/04/2012	Viernes 27/04/2012	Sabado 28/04/2012	Domingo 29/04/2012	Lunes 30/04/2012	Empleado	Operador	Tipo	Base	Con rutas reales	Con rutas imaginarias	Turno de mañana	Turno de tarde	Número de pernoctas	Días no asignados
-	T5	-	236	T5	-	-	ACP	N	CMTE	LPA	11	2	6	7	1	10
8538	8503	8200	-	135	8311	-	AMR	N	CMTE	LPA	15	1	8	8	2	11
107	-	107	8103	FS	FS	8214	ANA	N	CMTE	LPA	9	3	6	6	0	9
-	T6	171	T6	8145	-	167	APR	N	CMTE	LPA	10	4	6	8	0	13
-	135	-	8526	FS	FS	103	BLC	N	CMTE	LPA	15	2	8	9	1	13
T6	8512	-	-	510	8540	8503	CRR	N	CMTE	LPA	11	2	6	7	1	8
T5	8500	-	-	107	T5	-	DME	N	CMTE	LPA	10	6	8	8	0	13
-	-	236	512	FS	FS	8504	EAB	N	CMTE	LPA	15	1	9	7	1	14
8214	-	8538	8503	317	-	-	EAJ	N	CMTE	LPA	15	0	7	8	1	13
-	119	8165	8309	FS	FS	236	FDH	N	CMTE	LPA	8	3	5	6	0	11
518	8147	169	8540	501	8107	-	FGN	N	CMTE	LPA	12	2	7	7	1	11
-	-	101	-	590	8216	169	FMO	N	CMTE	LPA	10	1	5	6	0	12
8524	8538	8503	500	234	-	-	FSM	N	CMTE	LPA	15	1	7	9	2	14
VV	VV	VV	VV	VV	VV	VV	IVM	N	CMTE	LPA	10	2	6	6	0	10
-	GSS	T5	8506	FS	FS	8530	JBA	N	CMTE	LPA	12	2	6	8	1	14
8303	-	-	-	303	-	-	JRL	N	CMTE	LPA	13	3	8	8	1	14
8171	-	-	-	T6	T6	T6	JRP	N	CMTE	LPA	12	5	8	9	1	13
101	GSS	T6	224	FS	FS	107	JSM	N	CMTE	LPA	15	2	9	8	2	11
8503	8504	-	101	FS	FS	8538	JSN	N	CMTE	LPA	9	0	4	5	1,5	6
-	107	-	-	500	-	8303	JSP	N	CMTE	LPA	16	1	8	9	0	12
VV	VV	VV	VV	VV	VV	VV	MLG	N	CMTE	LPA	12	2	7	7	1	9
GSS	8236	-	T5	-	6100	-	OGF	N	CMTE	LPA	15	1	8	8	1	12
-	167	-	8135	FS	FS	8200	PCM	N	CMTE	LPA	14	3	8	9	1	13
500	101	8518	311	-	8147	-	RAM	N	CMTE	LPA	17	1	9	9	1	12
-	GSS	214	-	224	169	-	RHV	N	CMTE	LPA	12	2	7	7	1	12
236	8200	GSS	-	-	-	T5	SRS	N	CMTE	LPA	15	2	9	8	2	11
-	-	8303	8200	FS	FS	8147	VGS	N	CMTE	LPA	15	2	8	9	1	13
8212	-	-	-	8208	8226	-	RAN	N	CMTE	LPA	14	4	9	9	1	11

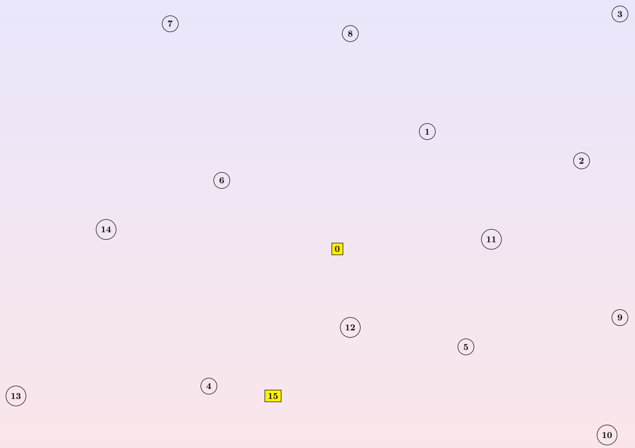


Figure: Instance n16-1-1-2-2a with 14 customers and 2 depots; at most 4 customers in a driver route; at most 7 customer in a vehicle route

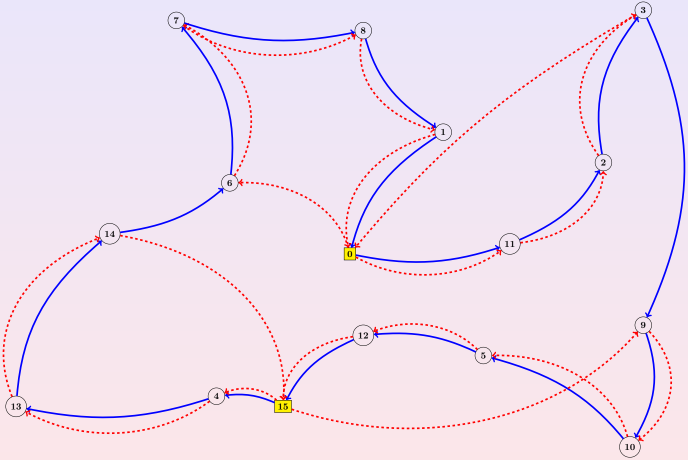


Figure: Optimal solution of n16-1-1-2-2a

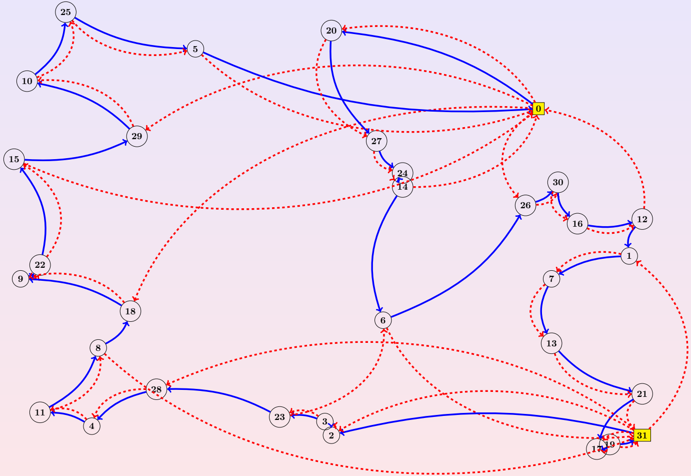


Figure: Optimal solution of n32-1-1-4-4a

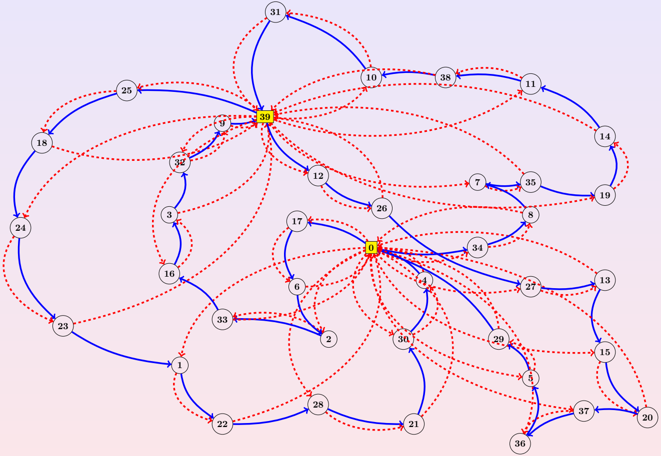


Figure: Optimal solution of n40-2-2-10-10a

Problem formulation

Notation:

- $V_c = \{1, \ldots, n\}$: set of *customers*. $V_d = \{0, n+1\}$: set of depots.
- $V = V_c \cup V_d$ vertices. $A = \{(i,j) : i,j \in V, i \neq j\}$ arcs. $G = (V, A)$ a directed graph.

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- c_{ij}^k: the cost to pay when a vehicle of type k traverses an arc (i,j), for $k \in \{1,2\}$.
- Q^k: the maximum number of customers that can be served by a vehicle of type k, for $k \in \{1,2\}$.
- K_d^k: the number of vehicles of type k available at the depot d, for $k \in \{1,2\}$.

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- For $S \subseteq V$: $\delta^+(S) := \{(i,j) \in A : i \in S,\ j \notin S\}$ and $A(S) := \{(i,j) \in A : i \in S,\ j \in S\}$.
- For $A' \subseteq A$: $x^k(A') = \sum_{(i,j)\in A'} x_{ij}^k$.

Problem formulation

Variables:

- $x_{ij}^{1} \in \{0,1\}$ takes value 1 iff a vehicle of type 1 (like crews) traverses arc $(i,j) \in A$.
- $x_{ij}^{2} \in \{0,1\}$ takes value 1 iff a vehicle of type 2 (like vehicles) traverses arc $(i,j) \in A$.
- $y_{ij} \in \{0,1\}$ takes value 1 iff a vehicle-change happens at (i,j).

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- $y_{ij} \in \{0,1\}$ takes value 1 iff a vehicle-change happens at (i,j).

- $z_i^k \geq 0$: the number of customers that a vehicle of type k has served immediately after serving customer i, with $k \in \{1,2\}$. (to control capacities)
- $w_i \geq 0$: the position in which customer i is served. (to control synchronization)

The problem can be modeled as follows:

$$\min \sum_{(i,j)\in A} c_{ij}^1 x_{ij}^1 + \sum_{(i,j)\in A} c_{ij}^2 x_{ij}^2 + N \sum_{i\in V_d, j\in V_c} x_{ij}^1 + M \sum_{(i,j)\in A} y_{ij}$$

The problem can be modeled as follows:

$$\min \sum_{(i,j)\in A} c^1_{ij}x^1_{ij} + \sum_{(i,j)\in A} c^2_{ij}x^2_{ij} + N \sum_{i\in V_d, j\in V_c} x^1_{ij} + M \sum_{(i,j)\in A} y_{ij}$$

subject to:

$$x^1(\delta^+(i)) = x^1(\delta^-(i)) = 1 \qquad \forall i \in V_c \tag{11}$$

$$x^1(\delta^+(j)) = x^1(\delta^-(j)) \leq K^1_j \qquad \forall j \in V_d \tag{12}$$

$$x^1(\delta^+(S)) \geq \sum_{i\in S}(x^1_{0,i} + x^1_{i,n+1}) \qquad \forall S \subseteq V_c : S \neq \emptyset \tag{13}$$

$$x^1(\delta^+(S)) \geq \sum_{i\in S}(x^1_{n+1,i} + x^1_{i,0}) \qquad \forall S \subseteq V_c : S \neq \emptyset \tag{14}$$

$$x^1_{ij} \in \{0,1\} \qquad \forall (i,j) \in A$$

$$x^2(\delta^+(i)) = x^2(\delta^-(i)) = 1 \qquad \forall i \in V_c \tag{15}$$

$$x^2(\delta^+(0)) = x^2(\delta^-(n+1)) \leq K_0^2 \tag{16}$$

$$x^2(\delta^+(n+1)) = x^2(\delta^-(0)) \leq K_{n+1}^2 \tag{17}$$

$$x^2(\delta^+(S)) \geq \sum_{i \in S}(x_{0,i}^2 + x_{i,0}^2) \qquad \forall S \subseteq V_c; S \neq \emptyset \tag{18}$$

$$x^2(\delta^+(S)) \geq \sum_{i \in S}(x_{n+1,i}^2 + x_{i,n+1}^2) \qquad \forall S \subseteq V_c; S \neq \emptyset \tag{19}$$

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$$x_{ij}^2 \in \{0,1\} \qquad \forall (i,j) \in A$$

$$z_j^k \geq z_i^k + x_{ij}^k - (Q^k - 1)(1 - x_{ij}^k) \qquad \forall (i,j) \in A; i,j \in V_c; k = 1,2 \tag{20}$$

$$w_j \geq w_i + x_{ij}^k - (n-1)(1 - x_{ij}^k) \qquad \forall (i,j) \in A; i,j \in V_c; k = 1,2 \tag{21}$$

$$x_{ij}^1 \leq x_{ij}^2 + y_{ij} \qquad \forall (i,j) \in A \tag{22}$$

$$y_{ij} \geq 0 \qquad \forall (i,j) \in A$$

$$w_i \geq 0 \qquad \forall i \in V_c$$

$$z_i^k \geq 0 \qquad \forall i \in V_c; k = 1,2.$$

Note 1: same inequalities in different format ($\geq$ versus $\leq$)

$$x^1(\delta^+(S)) \geq \sum_{i\in S}(x^1_{0,i} + x^1_{i,n+1}) \quad \equiv \quad x^1(0:S) + x^1(A(S)) + x^1(S:n+1) \leq |S|$$

$$x^1(\delta^+(S)) \geq \sum_{i\in S}(x^1_{n+1,i} + x^1_{i,0}) \quad \equiv \quad x^1(n+1:S) + x^1(A(S)) + x^1(S:0) \leq |S|$$

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Note 2: separating $x^1(\delta^+(S)) \geq \sum_{i\in S}(x^1_{0,i} + x^1_{i,n+1})$ for all $S \subseteq V_c$

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They can be rewritten as

$$x^1(\delta^+(S)) + \sum_{i\notin S}(x^1_{0,i} + x^1_{i,n+1}) \geq \sum_{i\in V_c}(x^1_{0,i} + x^1_{i,n+1}).$$

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Given x^*, let us consider a support graph $G' = (V', A')$ with $V' = V \cup \{s, t\}$, being s and t dummy nodes. The arc set A' is defined as follows:

- all the arcs $(i, j) \in A$ such that $x^{1*}_{ij} > 0$, each one with capacity x^{1*}_{ij},
- all arcs (s, i) with nodes $i \in V_c$, each one with capacity $x^{1*}_{0,i} + x^{1*}_{i,n+1}$, and
- the arcs (i, t) with nodes $i \in V_d$, each one with infinite capacity.

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Let $S \subset V'$ be the solution of the min-cut problem separating s from t, $s \in S$, in G'.

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- the arcs (i,t) with nodes $i \in V_d$, each one with infinite capacity.

Let $S \subset V'$ be the solution of the min-cut problem separating s from t, $s \in S$, in G'.

If the capacity of the arcs leaving S on G' is smaller than $\sum_{i\in V_c}(x^{1*}_{0,i} + x^{1*}_{i,n+1})$ then S defines a violated inequality (13) that must be added to the current linear program.

Valid inequalities

Subtour elimination inequalities

$$x^k(\delta^+(S)) \geq 1, \forall S \subseteq V_c, S \neq \emptyset, k \in V_d \quad (23)$$

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$$x^k(\delta^+(S)) \geq 1, \forall S \subseteq V_c, S \neq \emptyset, k \in V_d \tag{23}$$

Capacity inequalities

Fractional capacity constraints

$$x^k(\delta^+(S)) \geq |S|/Q^k, \forall S \subseteq V_c : S \neq \emptyset; k = 1, 2 \tag{24}$$

Rounded capacity constraints

$$x^k(\delta^+(S)) \geq \lceil |S|/Q^k \rceil, \forall S \subseteq V_c : S \neq \emptyset; k = 1, 2 \tag{25}$$

Multistar Inequalities (Gouveia, L., 1993)

$$x^k(\delta^+(S)) \geq \frac{1}{Q^k}(|S| + \sum_{i \in S} \sum_{j \notin S} (x_{ij}^k + x_{ji}^k)), \forall S \subseteq V_c, S \neq \emptyset, k = 1, 2 \qquad (26)$$

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D_p Inequalities (Grötschel, M. and Padberg, M. W., 1985)

Let $S = \{i_1, \ldots, i_p\} \subset V$ and a vehicle type k. Then, the D_p^- and D_p^+ inequality are given, respectively, by:

$$\sum_{j=1}^{p-1} x_{i_j,i_{j+1}}^k + x_{i_p,i_1}^k + 2\sum_{j=3}^{p} x_{i_1,i_j}^k + \sum_{j=4}^{p-1}\sum_{l=3}^{j-1} x_{i_j,i_l}^k \leq p-1 \tag{27}$$

$$\sum_{j=1}^{p-1} x_{i_j,i_{j+1}}^k + x_{i_p,i_1}^k + 2\sum_{j=2}^{p-1} x_{i_j,i_1}^k + \sum_{j=3}^{p-1}\sum_{l=2}^{j-1} x_{i_j,i_l}^k \leq p-1 \tag{28}$$

Strengthened Comb Inequalities (Lysgaard, J. et al., 2003)

$H \subset V_c$, $T_1, \dots, T_t \subset V_c$, $t \geq 2$ such that:

- $T_i \setminus H \neq \emptyset$ and $H \cap T_i \neq \emptyset$ for $i = 1, \dots, t$;
- for each $\{i, j\} \subset \{1, \dots, t\}$: $T_i \cap T_j \subset H$ or $T_i \cap T_j \cap H = \emptyset$.

$\forall S \subset V$, $\tilde{r}^k(S)$ equal to $\lceil d^k(S)/Q^k \rceil$ if $S \subset V_c$, and to $\lceil d^k(V_c \setminus S)/Q^k \rceil$ otherwise.

$$S(H, T_1, \dots, T_t) := \sum_{j=1}^{t} (\tilde{r}^k(T_j) + \tilde{r}^k(T_j \cap H) + \tilde{r}^k(T_j \setminus H))$$

If this quantity is odd, we consider that

$$x^k(\delta(H)) + \sum_{j=1}^{t} x^k(\delta(T_j)) \geq S(H, T_1, \dots, T_t) + 1 \tag{29}$$

is a *strengthened comb inequality*.

Cutting plane phase

- **Step 1:** Look for violated constraints (22).
- **Step 2:** Find violated constraints (26) (exact separation procedure) and (25) (heuristic separation procedure).
- **Step 3:** Apply a heuristic procedure to separate the inequalities (27) and (28).
- **Step 4:** Detect violated inequalities (13)–(14) and (18)–(19).
- **Step 5:** Look for violated inequalities (29).

Computational results

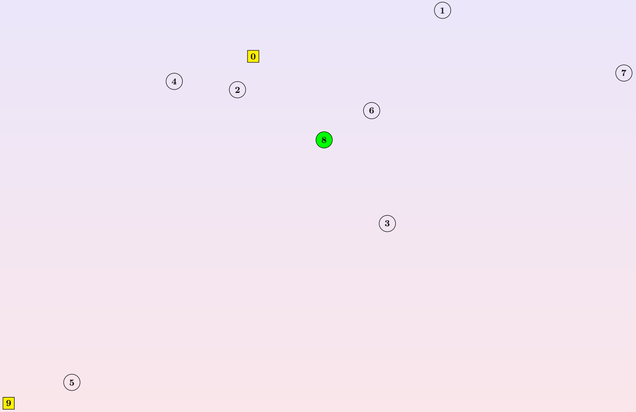
- Computer with Intel Core i3 CPU at 3.4 GHz and Cplex 12.6
- VRP instances have been properly adapted to suit our problem.
  - $n + 2 \in \{16, 20, 32, 40, 50\}$.
  - arc costs c_{ij} represent the Euclidean distance between the tasks (i, j).
  - Different values for M and N.
  - Time limit (T.L.) $= 2h$
- We created 69 instances of three classes:
  - Class a (small capacity):
  $$Q^k = \left\lceil \frac{n}{K_0^k + K_{n+1}^k} \right\rceil$$
  - Class b (medium capacity):
  $$Q^k = \left\lceil \left(\left\lceil \frac{n}{K_0^k + K_{n+1}^k} \right\rceil + (n+2) \right) /3 \right\rceil$$
  - Class c (large capacity):
  $$Q^k = n + 2$$

$M = 100,\ N = 10$								Alg 1 (z variables)								Alg 2 (capacity constraints)							
name	$n+2$	Q^2	Q^1	K_0^2	K_{n+1}^2	K_0^1	K_{n+1}^1	route	x	y	gap	r-time	time	nodes	cuts	route	x	y	gap	r-time	time	nodes	cuts
n16-1-1-2-2a	16	7	4	1	1	2	2	486	4	0	19.58	0.16	2279.52	64806	2084	486	4	0	10.09	0.09	64.23	3633	1053
n16-1-1-2-2b	16	8	7	1	1	2	2	443	2	0	8.64	0.14	10.53	914	302	443	2	0	7.78	0.06	2.89	275	343
n16-1-1-2-2c	16	16	16	1	1	2	2	411	2	0	1.86	0.12	0.19	4	121	411	2	0	2.47	0.08	0.12	4	126
n16-1-1-1-1a	16	7	7	1	1	1	1	447	2	0	9.21	0.14	13.92	1250	330	447	2	0	8.35	0.12	6.26	512	539
n16-1-1-1-1b	16	8	8	1	1	1	1	445	2	0	8.82	0.19	10.83	1040	325	445	2	0	8.44	0.14	7.00	530	688
n16-1-1-1-1c	16	16	16	1	1	1	1	411	2	0	1.62	0.20	0.25	4	126	411	2	0	2.23	0.08	0.14	8	131
n16-2-2-1-1a	16	4	7	2	2	1	1	507	2	2	8.73	0.14	250.65	16933	752	507	2	2	7.42	0.14	179.96	7923	1559
n16-2-2-1-1b	16	7	8	2	2	1	1	481	2	2	6.32	0.11	18.94	2223	382	481	2	2	5.94	0.12	11.90	1423	618
n16-2-2-1-1c	16	16	16	2	2	1	1	454	2	2	2.63	0.11	0.27	32	216	454	2	2	2.63	0.12	0.39	70	233
n16-2-2-4-4a	16	4	2	2	2	4	4	626	7	0	25.72	0.19	1266.24	32829	1762	626	7	0	7.69	0.25	131.34	6084	1276
n16-2-2-4-4b	16	7	6	2	2	4	4	497	4	0	3.72	0.19	3.00	269	261	497	4	0	5.06	0.25	2.39	228	349
n16-2-2-4-4c	16	16	16	2	2	4	4	484	4	0	3.63	0.31	0.50	20	192	484	4	0	3.62	0.20	0.34	25	207
n20-1-1-2-3a	20	9	4	1	1	2	3	675	5	1	41.09	0.11	T.L.	32379	7530	678	5	1	30.25	0.30	T.L.	20808	11061
n20-1-1-2-3b	20	10	8	1	1	2	3	531	3	0	13.50	0.14	5020.60	67709	3010	531	3	0	12.12	0.19	1238.29	16878	3301
n20-1-1-2-3c	20	20	20	1	1	2	3	503	2	0	7,07	0.11	4.27	204	343	503	2	0	8.14	0.12	4.99	377	471
n20-1-1-3-3a	20	9	3	1	1	3	3	666	6	0	33.06	0.11	T.L.	37032	5482	652	6	0	11.42	0.30	T.L.	38704	6033
n20-1-1-3-3b	20	10	8	1	1	3	3	531	3	0	13.50	0.17	4056.32	59987	2776	531	3	0	12.12	0.19	2124.20	21108	4192
n20-1-1-3-3c	20	20	20	1	1	3	3	503	2	0	7.07	0.11	3.99	193	343	503	2	0	8.14	0.12	6.55	506	537
n20-2-2-5-5a	20	5	2	2	2	5	5	860	10	0	40.10	0.20	T.L.	25832	6582	861	10	0	16.19	0.47	T.L.	25228	8905
n20-2-2-5-5b	20	9	8	2	2	5	5	589	4	0	8.59	0.17	191.91	7088	780	589	4	0	9.10	0.30	202.27	6168	1333
n20-2-2-5-5c	20	20	20	2	2	5	5	554	4	0	5.74	0.19	2.65	150	307	554	4	0	5.60	0.16	2.68	214	370
n20-5-5-2-2a	20	2	5	5	5	2	2	939	4	6	11.22	0.27	T.L.	71429	2346	951	4	6	7.17	1.01	T.L.	23951	7107
n20-5-5-2-2b	20	8	9	5	5	2	2	787	4	6	3.05	0.22	21.98	1735	466	787	4	6	1.62	0.25	8.99	917	576
n20-5-5-2-2c	20	20	20	5	5	2	2	771	4	6	1.74	0.20	1.34	94	374	771	4	6	0.92	0.30	0.62	42	377

Alg 3 (all valid inequalities)								$M = 100,\ N = 10$								$M = 10,\ N = 1$							
name	$n+2$	Q^2	Q^1	K_0^2	K_{n+1}^2	K_0^1	K_{n+1}^1	route	x	y	gap	r-time	time	nodes	cuts	route	x	y	gap	r-time	time	nodes	cuts
n16-1-1-2-2a	16	7	4	1	1	2	2	486	4	0	3.92	0.16	0.87	69	443	486	4	0	2.03	0.19	0.95	50	235
n16-1-1-2-2b	16	8	7	1	1	2	2	443	2	0	3.56	0.09	0.73	51	358	443	2	0	3.71	0.11	1.06	80	463
n16-1-1-2-2c	16	16	16	1	1	2	2	411	2	0	0.46	0.17	0.19	4	136	397	1	1	0.25	0.09	0.09	3	93
n16-1-1-1-1a	16	7	7	1	1	1	1	447	2	0	3.21	0.19	2.95	136	1016	447	2	0	3.34	0.16	2.43	140	598
n16-1-1-1-1b	16	8	8	1	1	1	1	445	2	0	3.87	0.06	1.19	98	396	445	2	0	6.04	0.06	2.43	169	652
n16-1-1-1-1c	16	16	16	1	1	1	1	411	2	0	0.23	0.20	0.22	3	137	397	1	1	0.25	0.11	0.12	3	89
n16-2-2-1-1a	16	4	7	2	2	1	1	507	2	2	2.96	0.16	2.98	174	870	507	2	2	3.69	0.39	2.87	143	418
n16-2-2-1-1b	16	7	8	2	2	1	1	481	2	2	2.50	0.08	0.66	78	383	481	2	2	5.17	0.06	2.04	168	668
n16-2-2-1-1c	16	16	16	2	2	1	1	454	2	2	0.67	0.09	0.11	4	224	440	1	3	0.21	0.08	0.11	4	103
n16-2-2-4-4a	16	4	2	2	2	4	4	626	7	0	0.44	0.41	0.58	12	273	626	7	0	0.57	0.20	0.73	33	190
n16-2-2-4-4b	16	7	6	2	2	4	4	497	4	0	3.72	0.16	0.62	47	449	484	3	1	2.11	0.11	1.03	49	323
n16-2-2-4-4c	16	16	16	2	2	4	4	484	4	0	1.34	0.20	0.31	18	232	440	1	3	0.21	0.12	0.12	3	97
n20-1-1-2-3a	20	9	4	1	1	2	3	624	5	1	16.37	0.69	T.L.	20324	10269	613	5	1	5.39	0.78	37.39	982	917
n20-1-1-2-3b	20	10	8	1	1	2	3	531	3	0	4.46	0.33	9.81	392	792	529	4	0	4.80	0.42	7.24	317	544
n20-1-1-2-3c	20	20	20	1	1	2	3	503	2	0	7.07	0.11	3.88	225	555	466	1	1	1.89	0.09	0.16	6	61
n20-1-1-3-3a	20	9	3	1	1	3	3	651	6	0	1.84	0.75	4.56	124	561	651	6	0	2.44	0.45	1.84	61	328
n20-1-1-3-3b	20	10	8	1	1	3	3	531	3	0	4.46	0.33	6.22	252	883	529	4	0	5.16	0.27	15.02	590	998
n20-1-1-3-3c	20	20	20	1	1	3	3	503	2	0	7.07	0.14	3.90	225	555	466	1	1	1.89	0.08	0.14	6	61
n20-2-2-5-5a	20	5	2	2	2	5	5	821	10	0	4.23	0.78	140.24	3679	1292	803	10	1	2.83	0.69	17.10	595	544
n20-2-2-5-5b	20	9	8	2	2	5	5	589	4	0	3.97	0.41	7.35	291	1084	572	3	1	2.91	0.31	2.22	113	418
n20-2-2-5-5c	20	20	20	2	2	5	5	554	4	0	3.20	0.20	0.87	47	304	529	3	1	0.55	0.17	0.20	4	207
n20-5-5-2-2a	20	2	5	5	5	2	2	927	4	6	1.24	1.61	36.16	781	1181	907	4	7	1.16	1.83	11.37	264	810
n20-5-5-2-2b	20	8	9	5	5	2	2	787	4	6	0.88	0.33	1.42	89	531	777	3	7	1.53	0.28	2.37	120	563
n20-5-5-2-2c	20	20	20	5	5	2	2	771	4	6	0.35	0.39	0.51	16	427	761	3	7	0.48	0.20	0.28	11	213

name	$n+2$	Q^2	Q^1	K_0^2	K_{n+1}^2	K_0^1	K_{n+1}^1	$M=100,\ N=10$ route	x	y	gap	r-time	time	nodes	cuts	$M=10,\ N=1$ route	x	y	gap	r-time	time	nodes	cuts
n32-1-1-4-4a	32	15	4	1	1	4	4	1566	8	0	4.27	5.54	T.L.	25290	4489	1534	8	1	3.39	4.96	491.48	3958	2111
n32-1-1-4-4b	32	16	12	1	1	4	4	1155	4	0	8.10	6.86	T.L.	18266	13555	1098	3	2	3.28	5.29	416.10	4002	2389
n32-1-1-4-4c	32	32	32	1	1	4	4	979	2	0	4.7	2.06	73.68	1081	1616	913	1	1	0.00	1.73	1.73	1	535
n32-2-2-1-1a	32	8	15	2	2	1	1	1269	2	2	3.80	10.51	1540.79	4907	9962	1216	2	6	4.29	4.27	1183.05	10214	2510
n32-2-2-1-1b	32	14	16	2	2	1	1	1170	2	2	1.97	15.68	197.86	1374	2612	1149	2	3	1.86	6.10	144.68	1342	3556
n32-2-2-1-1c	32	32	32	2	2	1	1	1053	2	2	0.00	6.40	6.40	1	1065	1038	1	3	0.00	1.87	1.89	1	521
n32-2-2-5-5a	32	8	3	2	2	5	5	1970	10	2	11.06	10.75	T.L.	16075	5803	1920	10	4	2.45	10.02	974.30	6729	1955
n32-2-2-5-5b	32	14	12	2	2	5	5	1281	4	0	6.60	8.85	3592.36	16702	7943	1203	3	4	3.47	4.02	700.27	6506	3271
n32-2-2-5-5c	32	32	32	2	2	5	5	1139	4	0	2.63	2.01	15.71	283	1017	1038	1	3	0.16	2.42	3.06	11	585
n40-2-2-10-10a	40	10	2	2	2	10	10	1374	19	0	1.19	14.38	1512.37	6498	3457	1374	19	0	1.48	15.62	1343.12	6508	2789
n40-2-2-10-10b	40	17	14	2	2	10	10	824	4	0	6.57	7.97	T.L.	11663	12594	769	3	3	3.67	7.96	4915.97	21148	8594
n40-2-2-10-10c	40	40	40	2	2	10	10	791	4	0	3.43	8.08	2754.65	12835	4813	748	2	2	0.65	4.17	17.05	188	880
n40-4-4-4-4a	40	5	5	4	4	4	4	1203	8	0	9.57	29.80	T.L.	9192	7810	1252	8	4	15.73	41.93	T.L.	10533	7921
n40-4-4-4-4b	40	15	15	4	4	4	4	946	8	0	3.75	22.89	T.L.	8847	14005	902	6	2	2.37	7.16	1522.34	6769	7199
n40-4-4-4-4c	40	40	40	4	4	4	4	938	8	0	3.00	19.69	5840.10	16343	4519	849	3	5	0.17	6.35	7.47	8	752
n40-10-10-2-2a	40	2	10	10	10	2	2	1560	4	16	0.34	81.26	1242.30	2581	3471	1560	4	16	0.59	63.99	1067.11	2822	3503
n40-10-10-2-2b	40	14	17	10	10	2	2	1326	4	16	0.37	19.02	590.00	2172	4282	1326	4	16	0.83	13.03	457.77	3158	3389
n40-10-10-2-2c	40	40	40	10	10	2	2	1310	4	16	0.10	11.72	12.53	6	1828	1299	3	17	0.07	6.32	7.22	10	708
n50-6-6-2-2a	50	4	12	6	6	2	2	1464	4	8	8.93	204.42	T.L.	3766	7773	1343	4	8	6.03	116.22	T.L.	6328	9525
n50-6-6-2-2b	50	18	21	6	6	2	2	1077	4	8	0.49	36.99	918.38	1521	5318	1077	4	8	0.81	41.11	429.24	1575	4648
n50-6-6-2-2c	50	50	50	6	6	2	2	1065	4	8	0.03	18.63	19.44	11	2699	1065	4	8	0.09	15.32	16.89	9	941
n50-8-8-2-2a	50	3	12	8	8	2	2	1643	4	12	6.18	311.30	T.L.	3286	10281	1578	4	18	10.08	210.60	T.L.	6423	10364
n50-8-8-2-2b	50	18	21	8	8	2	2	1214	4	12	0.41	29.33	715.12	1333	5389	1214	4	12	0.75	29.05	219.31	1050	2429
n50-8-8-2-2c	50	50	50	8	8	2	2	1202	4	12	0.04	28.20	38.58	39	3330	1202	4	12	0.08	23.40	27.02	22	1360
n50-8-8-8-8a	50	3	3	8	8	8	8	1958	16	0	2.85	308.82	T.L.	4315	6957	1930	16	0	3.83	149.85	T.L.	8803	4218
n50-8-8-8-8b	50	18	18	8	8	8	8	1420	16	0	2.03	46.24	6275.34	10041	5821	1238	7	10	1.34	21.04	1515.88	4889	7362
n50-8-8-8-8c	50	50	50	8	8	8	8	1415	16	0	1.71	29.76	1451.36	3909	3556	1219	6	10	0.08	22.59	25.80	14	1203

1. Motivation: case study in air transport
   - Need of optimization!
   - Crew-and-aircraft routing
   - Crew-and-aircraft rostering
2. The Vehicle and Driver Scheduling Problem
   - Examples of solution
   - Problem formulation
   - Valid inequalities & Cutting plane phase
   - Computational results
3. The Driver and Vehicle Routing Problem
   - Examples of solution
   - Problem Formulation
   - Valid Inequalities & Cutting plane phase
   - Computational results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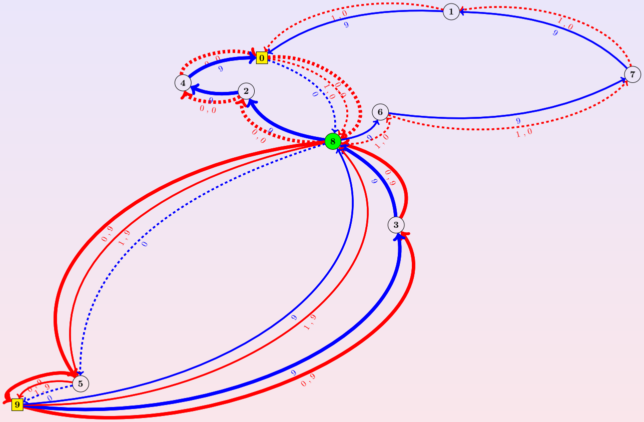

Figure: Solution examples of the instance n10-2 with $T = 5$ and $|V_e| = 1$

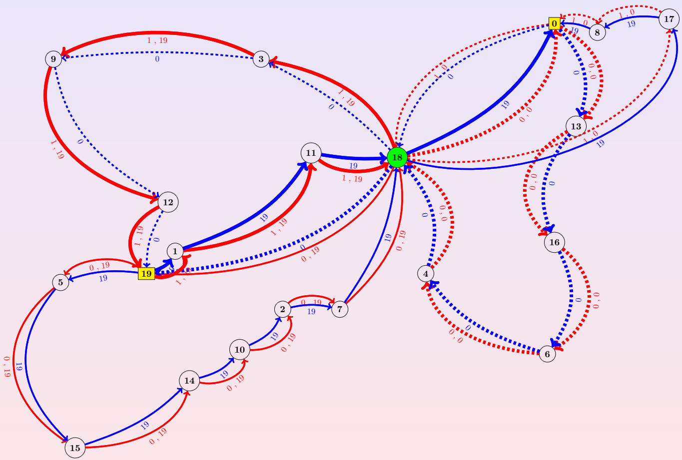

Figure: Solution examples of the instance n20-4 with $T = 7$ and $|V_e| = 1$

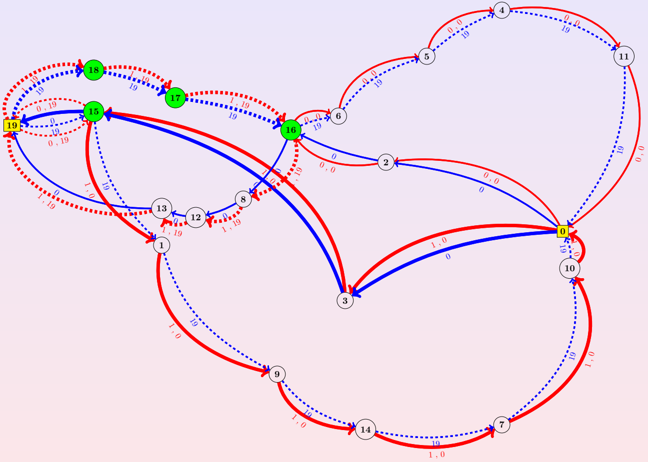

Figure: Solution examples of the instance n20-5 with $T = 5$ and $|V_e| = 4$

Problem Formulation

Notation:

Given:

- $V_d = \{0, n+1\}$: set of depots. $V_c = \{1, \ldots, n\}$: set of customer locations. $V_c = V_r \cup V_e$. V_r: set of *regular* customer locations. V_e: set of *exchange* locations.
- $V = V_d \cup V_r \cup V_e$: the vertex set. $A = \{(i,j) : i,j \in V, i \neq j\}$: set of potential arcs. $G = (V, A)$ is a directed graph.
- c_{ij}: cost to pay when a driver traverses an arc (i,j).
- K_d: set of drivers available at each depot $d \in D$. In total $K = \sum_{d \in D} K_d$ drivers.
- L_d: set of vehicles available at each depot $d \in D$. In total $L = \sum_{d \in D} L_d$ vehicles.
- t_{ij}: time a pair vehicle-driver spends to traverse an arc (i,j).
- T: maximum time that a driver can spend performing a single route.

Problem formulation

Variables:

- x_{ij}^k takes value 1 if the driver $k \in K$ traverses the arc $(i,j) \in A$, and 0 otherwise.
- y_{ij}^d represents the number of vehicles that start their routes from depot d and traverse an arc $(i,j) \in A$.
- u_i^k takes value 1 if the driver $k \in K$ visits the customer i, and 0 otherwise.
- v_i^d represents the number of vehicles originating from depot $d \in D$ and visiting customer i.
- w_i represents the (ordered) position in which customer i is served.

The DVRP can be modeled as follows:

$$\min \sum_{k\in K} \sum_{(i,j)\in A} c_{ij} x_{ij}^k \tag{30}$$

subject to:

$$x^k(\delta^+(i)) = x^k(\delta^-(i)) = u_i^k \qquad i \in V_c, k \in K \tag{31}$$

$$\sum_{k\in K} u_i^k \geq 1 \qquad i \in V_c \tag{32}$$

$$x^k(\delta^+(n+1)) = x^k(\delta^-(n+1)) = 0 \qquad k \in K_0 \tag{33}$$

$$x^k(\delta^+(0)) = x^k(\delta^-(0)) = 0 \qquad k \in K_{n+1} \tag{34}$$

$$\sum_{(i,j)\in A} t_{ij} x_{ij}^k \leq T \qquad k \in K \tag{35}$$

$$w_j \geq w_i + x_{ij}^k - (n-1)(1 - x_{ij}^k) \qquad k \in K, (i,j) \in A,\ i,j \in V_c \tag{36}$$

$$w_i \geq 0 \qquad i \in V_c \tag{37}$$

$$x_{ij}^k \in \{0,1\} \qquad k \in K, (i,j) \in A \tag{38}$$

$$u_i^k \in \{0,1\} \qquad i \in V, k \in K. \tag{39}$$

$$y^d(\delta^+(i)) = y^d(\delta^-(i)) = v_i^d \qquad i \in V_c, d \in D \tag{40}$$

$$\sum_{d \in D} v_i^d = 1 \qquad i \in V_r \tag{41}$$

$$\sum_{d \in D} v_i^d \geq 1 \qquad i \in V_e \tag{42}$$

$$y^0(\delta^+(n+1)) = y^0(\delta^-(0)) = 0 \tag{43}$$

$$y^{n+1}(\delta^+(0)) = y^{n+1}(\delta^-(n+1)) = 0 \tag{44}$$

$$1 \leq y^0(\delta^+(0)) = y^0(\delta^-(n+1)) \leq |L_0| \tag{45}$$

$$1 \leq y^{n+1}(\delta^+(n+1)) = y^{n+1}(\delta^-(0)) \leq |L_{n+1}| \tag{46}$$

$$w_j \geq w_i + y_{ij}^d - (n-1)(1 - y_{ij}^d) \qquad d \in D, (i,j) \in A : \; i,j \in V_c \tag{47}$$

$$y_{ij}^d \geq 0 \qquad d \in D, (i,j) \in A \tag{48}$$

$$v_i^d \geq 0 \qquad i \in V_c, d \in D. \tag{49}$$

$$y^d(\delta^+(i)) = y^d(\delta^-(i)) = v_i^d \qquad i \in V_c, d \in D \tag{40}$$

$$\sum_{d \in D} v_i^d = 1 \qquad i \in V_r \tag{41}$$

$$\sum_{d \in D} v_i^d \geq 1 \qquad i \in V_e \tag{42}$$

$$y^0(\delta^+(n+1)) = y^0(\delta^-(0)) = 0 \tag{43}$$

$$y^{n+1}(\delta^+(0)) = y^{n+1}(\delta^-(n+1)) = 0 \tag{44}$$

$$1 \leq y^0(\delta^+(0)) = y^0(\delta^-(n+1)) \leq |L_0| \tag{45}$$

$$1 \leq y^{n+1}(\delta^+(n+1)) = y^{n+1}(\delta^-(0)) \leq |L_{n+1}| \tag{46}$$

$$w_j \geq w_i + y_{ij}^d - (n-1)(1 - y_{ij}^d) \qquad d \in D, (i,j) \in A: \ i,j \in V_c \tag{47}$$

$$y_{ij}^d \geq 0 \qquad d \in D, (i,j) \in A \tag{48}$$

$$v_i^d \geq 0 \qquad i \in V_c, d \in D. \tag{49}$$

$$\sum_{k \in K} x_{ij}^k \geq \sum_{d \in D} y_{ij}^d \qquad (i,j) \in A \tag{50}$$

$$x_{ij}^k \leq \sum_{d \in D} y_{ij}^d \qquad k \in K, (i,j) \in A. \tag{51}$$

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No-change constraints

$$x^k(S : V_c \setminus S) \geq x^k(d : S) \qquad d \in D, k \in K_d, S \subseteq V_r \tag{55}$$

Note 3: $x^k(S : V_c \setminus S) \geq x^k(d : S)$ for all $d \in D, k \in K_d, S \subseteq V_r$

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They can be rewritten as

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Given x^*, let us consider two dummy nodes s and t, and define a graph $G' = (V', A')$ where the node set is $V' = V_c \cup \{s, t\}$ and the arc set A' includes

- all the arcs $(i, j) \in A$ with $i, j \in V_c$, each one with capacity x_{ij}^{*k},
- a new arc (s, i) for each $i \in V_r$, with capacity x_{di}^{*k}, and
- a new arc (i, t) for each $i \in V_e$, with infinite capacity.

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Let $S' \subset V'$ be the optimal min-cut solution separating s from t in G', with $d \in S'$. Note that the set S' contains only the depot d and some customers in V_r.

If the capacity of the arcs leaving S' is smaller than $x^{*k}(d : V_r)$, then $S = S' \setminus \{d\}$ defines a violated inequality (55) to add to the current linear program.

Cutting plane phase

- **Step 1:** To detect violated constraints (54).
- **Step 2:** To find violated constraints (55).
- **Step 3:** To look for violated constraints (50).
- **Step 4:** To find violated inequalities (51).

Computational results

- Computer with Intel Core i3 CPU at 3.4 GHz
- Cplex 12.7
- Random instances with $n+2 \in \{10, 15, 20, 25, 30\}$. Coordinates in $[0, 100] \times [0, 100]$
- The arc costs c_{ij} are the Euclidean distance between i and j
- $|V_e| = 1$
- $t_{ij} = c_{ij}/60 + 60$ for all $(i, j) \in A$
- Four values for T on each instance: T_A, T_B, T_C, T_D
- $K_d = L_d = 3$ for all $d \in D$
- Time limit (T.L.) $= 2$ hours.

			(30)–(53)		(30)–(55)		(30)–(53), (54)		(30)–(54)	
name	$n+2$	T_C	time	GAP	time	GAP	time	GAP	time	GAP
n10-1	10	8	34.24	33.66	2.28	16.59	7.89	23.78	0.73	5.08
n10-2	10	7	2.00	9.25	0.23	4.11	0.31	3.60	0.08	0.00
n10-3	10	7	20.11	33.59	0.87	1.68	4.59	25.26	0.14	1.08
n10-4	10	7	10.53	18.75	0.41	11.46	0.53	8.94	0.27	0.18
n10-5	10	7	18.52	34.83	1.00	2.53	5.52	24.82	0.12	1.02
n15-1	15	9	532.42	30.37	129.03	27.79	17.55	13.75	0.97	6.94
n15-2	15	9	87.30	24.14	0.36	0.00	21.86	20.94	0.23	0.00
n15-3	15	9	138.09	17.27	5.49	0.96	6.80	13.90	0.19	0.00
n15-4	15	9	1390.84	21.93	125.07	13.01	47.14	16.28	0.69	1.15
n15-5	15	9	T.L.	37.37	17.67	10.10	130.56	20.34	0.59	0.18
n20-1	20	10	T.L.	67.08	6877.34	13.64	T.L.	26.48	10.41	6.00
n20-2	20	10	T.L.	24.66	1433.32	11.89	188.96	17.86	9.63	4.61
n20-3	20	10	T.L.	56.83	T.L.	16.11	1886.21	19.34	32.84	4.56
n20-4	20	10	T.L.	14.99	295.95	9.17	53.73	11.62	9.66	3.96
n20-5	20	10	T.L.	23.79	T.L.	18.59	485.27	17.36	7.78	5.72
n25-1	25	11	T.L.	47.54	T.L.	8.42	591.27	8.52	110.18	1.53
n25-2	25	11	T.L.	61.30	T.L.	53.79	4722.38	17.51	167.98	6.53
n25-3	25	11	T.L.	20.08	4033.76	8.88	60.92	10.12	9.87	2.93
n25-4	25	11	T.L.	47.21	1479.67	13.65	581.18	13.39	22.28	1.70
n25-5	25	11	T.L.	28.68	3852.10	14.87	T.L.	52.45	58.72	5.54
n30-1	30	13	T.L.	61.22	T.L.	47.76	T.L.	47.14	135.27	6.11
n30-2	30	13	T.L.	67.56	T.L.	50.41	T.L.	15.89	62.06	1.25
n30-3	30	13	T.L.	43.40	T.L.	14.86	T.L.	22.47	38.78	3.46
n30-4	30	13	T.L.	57.35	T.L.	48.39	778.87	12.41	146.89	7.75
n30-5	30	13	T.L.	33.85	448.96	3.33	T.L.	14.57	18.35	1.03

name	$n+2$	T_A	GAP	sol	time	T_B	GAP	sol	time	T_C	GAP	sol	time	T_D	GAP	sol	time
n10-1	10	6	34.83	652	7.22	7	8.53	442	0.41	8	5.08	410	0.73	10	0.00	369	0.05
n10-2	10	5	38.32	486	17.43	6	0.00	292	0.06	7	0.00	292	0.08	10	0.00	292	0.05
n10-3	10	5	56.69	987	272.77	6	39.95	646	327.38	7	1.08	390	0.14	10	0.00	383	0.06
n10-4	10	5	33.21	610	4.52	6	28.15	534	8.00	7	0.18	384	0.27	10	0.00	383	0.06
n10-5	10	5	35.39	595	6.91	6	2.88	365	0.59	7	1.02	356	0.12	10	0.00	350	0.06
n15-1	15	6	22.35	454	34.43	8	4.58	349	0.83	9	6.94	349	0.97	12	0.00	302	0.23
n15-2	15	6	34.19	746	60.59	8	1.67	414	1.06	9	0.00	406	0.23	12	0.00	406	0.25
n15-3	15	6	33.95	660	50.44	8	12.14	442	9.72	9	0.00	388	0.19	12	0.00	388	0.20
n15-4	15	6	46.94	1094	T.L.	8	28.71	715	473.82	9	1.15	497	0.69	12	1.29	460	1.37
n15-5	15	6	30.61	787	83.09	8	37.31	751	T.L.	9	0.18	471	0.59	12	0.00	469	0.31
n20-1	20	7	54.47	1285	T.L.	9	36.32	846	T.L.	10	6.00	557	10.41	14	0.74	520	2.48
n20-2	20	7	29.72	666	4071.75	9	4.69	450	4.84	10	4.61	438	9.63	14	2.85	399	2.06
n20-3	20	7	32.15	845	6576.33	9	30.13	757	T.L.	10	4.56	540	32.84	14	0.57	507	2.82
n20-4	20	7	23.40	600	997.44	9	25.14	580	6305.20	10	3.96	447	9.66	14	0.00	415	0.44
n20-5	20	7	30.14	647	3623.64	9	22.32	538	1574.83	10	5.72	432	7.78	14	8.31	409	13.71
n25-1	25	8	25.41	718	2002.73	10	29.36	711	T.L.	11	1.53	499	110.18	16	0.00	483	4.57
n25-2	25	8	39.53	825	T.L.	10	33.71	716	T.L.	11	6.53	501	167.98	16	5.36	483	52.12
n25-3	25	8	24.19	603	3463.28	10	22.07	555	4473.84	11	2.93	439	9.87	16	0.00	405	1.31
n25-4	25	8	26.39	683	3530.49	10	30.85	680	T.L.	11	1.70	469	22.28	16	0.22	454	9.50
n25-5	25	8	28.77	721	3720.39	10	32.62	703	T.L.	11	5.54	491	58.72	16	0.00	451	2.87
n30-1	30	9	26.98	877	T.L.	12	31.76	864	T.L.	13	6.11	615	135.27	18	2.58	581	57.35
n30-2	30	9	33.64	927	T.L.	12	35.18	863	T.L.	13	1.25	560	62.06	18	0.00	552	22.12
n30-3	30	9	31.68	812	T.L.	12	40.06	818	T.L.	13	3.46	506	38.78	18	0.00	485	10.41
n30-4	30	9	29.10	756	T.L.	12	25.94	676	T.L.	13	7.75	536	146.89	18	2.60	495	121.43
n30-5	30	9	29.67	756	T.L.	12	33.90	739	T.L.	13	1.03	493	18.35	18	0.97	490	35.32

Summarizing ...

- While working on Autonomous Vehicle Routing is important today (supported by EU projects), we cannot forget that in the coming years both vehicles and drivers still coexists.
- Vehicles and humans have different constraints when designing their duties (routes).
- In many situations the integrated vehicle-driver problem is the combination of two VRPs with some linking variables.
- All the experiences and advances on VRPs help to produce better solutions in real-world applications.
- We have shown that optimal solutions for an integrated vehicle-driver problem can be generated for a regional airline, and then perhaps for other companies too!
- We have described two academic vehicle-driver problems for which optimal solutions can also be generated. Other variants are interesting research challenges for future investigations.

References

- J-J. Salazar–González. Approaches to solve the fleet-assignment, aircraft-routing, crew-pairing and crew-rostering problems of a regional carrier. Omega 43, 71–82, 2014.
- V. Cacchiani, J-J. Salazar–González. Optimal solutions to a real-world integrated airline scheduling problem. Transportation Science 51, 250–268, 2017.
- V. Cacchiani, J-J. Salazar–González. Heuristic approaches for flight retiming in an integrated airline scheduling problem of a regional carrier. Omega, (to appear), 2019.
- B. Domínguez–Martín, I. Rodríguez–Martín, and J-J. Salazar–González. An exact algorithm for a vehicle-and-driver scheduling problem. Computers and Operations Research 81, 247–256, 2017.
- B. Domínguez–Martín, I. Rodríguez–Martín, and J-J. Salazar–González. The driver and vehicle routing problem. Computers and Operations Research 92, 56–64, 2018.
- B. Domínguez–Martín, I. Rodríguez–Martín, and J-J. Salazar–González. A heuristic approach to the driver and vehicle routing problem. R. Cerulli et al. (Eds.): ICCL 2018, LNCS 11184, 295–305, 2018.

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2011	0.765	24	42/77
2012	0.843	40	44/79
2013	0.766	25	54/79
2014	0.831	57	61/81
2015	0.927	32	55/82
2016	0.899	32	64/83
2017	**1.094**	25	60/83
2018	?	25	?